ABSTRACT

Multi-Parameter Functions in Chaotic Dynamical Systems

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For two semesters, a fellow math major and I thoroughly proved results from Sections 1.1 – 1.8 of An Introduction to Chaotic Dynamical Systems by Robert Devaney. After going through Devaney's calculations and proofs, I created a multi-parameter family of functions to consider and observe. This is a piecewise function of polynomials that always intersects the x-axis at 0 and 1. It has two maxima and one minimum value. Depending on the range of the parameters, the minimum value can be above or below the x-axis. I have analyzed its behavior and determined the fixed and periodic points. I found that at certain parameter values the family of function's corresponding invariant set will be closed and totally disconnected. I conjecture that the invariant set is a perfect subset of the unit interval which would make it a Cantor set. Next, if the same parameter values could be used to show the new equation maps are chaotic. Dr. Brian Raines will guide me through the steps of this process.

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ACKNOWLEDGMENTS

This thesis would not have been possible without the help of many people. I am so thankful to have had the opportunity to learn from Dr. Brian Raines, my thesis advisor. This project sparked my desire to pursue a Ph.D. that will use my mathematics background after graduation. Without the encouragement from inspiring professors like Dr. Raines and Dr. Johnny Henderson I would never have dreamed this possible. I am also grateful to my family and friends, who supported me on this journey. Specifically, I would like to thank Emily Joslin, the most encouraging and supportive study buddy and best friend I could have ever asked for. Thank you.

CHAPTER ONE

Background Material and Proofs

This chapter includes definitions of necessary terms and notations from calculus and topology. This information will be used to prove the stated theorems. This is simply a selection of the material covered in the first year of research which I thought was most beneficial and applicable to my project.

1.1 Elementary Definitions

Functions come in all different shapes and sizes, but over the years, mathematicians have defined characteristics which we can use to compare and contrast functions. These definitions allow us to distinguish functions by more than their outputs. To consider the family of all functions is very broad and will not lead to a significant conclusion. Some of these characteristics are necessary assumptions in the following impactful theorems.

To begin we will denote the set of natural numbers with $\mathbb{N} = 1, 2, 3, 4...$ and the set of real numbers with $\mathbb{R}$. The reals include rational numbers, irrational numbers, the entire set of the naturals and more. The reals do not include imaginary numbers or positive or negative infinity. Finally let the set $I := \{x \in \mathbb{R} | 0 \leq x \leq 1\}$.

Definition 1.1. Let f be a function that maps I into J and consider $x, y \in I$. $f(x)$ is *one-to-one* if $f(x) \neq f(y)$ whenever $x \neq y$.

Definition 1.2. Let I and J be intervals and $f : I \to J$. The function f is *onto* if for any y in J there is an $x \in I$ such that $f(x) = y$.

Definition 1.3. Let f be a function that maps E into Y. Let $p \in E$. The function f is *continuous* at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all points in E for which $d_X(x, p) < \delta$.

Definition 1.4. Let $f : I \to J$. The function $f(x)$ is a *homeomorphism* if $f(x)$ is one-to-one, onto, and continuous, and $f^{-1}(x)$ is also continuous.

A one-to-one function is not always onto and an onto function is not always one-to-one. However, a homeomorphism is both one-to-one and onto. In the next couple of definitions, we will characterize a set of points. Sets can be empty, i.e. the empty set, or contain an infinite number of points. Also note that the *complement* of a set $A \subset X$ is the set of all points in X but not in A or $A' = \{x \in X | x \notin A\}$.

Definition 1.5. Let $S \subset \mathbf{R}$. S is an *open set* if, for any $x \in S$, there is an $\epsilon > 0$ such that all points t in the open interval $x - \epsilon < t < x + \epsilon$ are contained in S.

Definition 1.6. Let $S \subset \mathbf{R}$. A point $x \in \mathbf{R}$ is a *limit point* of S if there is a sequence of points $x_n \in S$ converging to x. S is a *closed set* if it contains all of its limit points.

Definition 1.7. Let $S \subset \mathbf{R}$. A subset U of S is *dense* in S if $\overline{U} = S$.

Next, points will be characterized by how they relate to certain functions and sets.

Definition 1.8. Let $f : I \to J$ and consider $x \in I$. The point x is a *fixed point* for f if $f(x) = x$.

Definition 1.9. Let $f : I \to J$ and consider $x \in I$. The point x is a *periodic point* of period n if $f^n(x) = x$. We denote the *set of periodic points* of period n by $Per_n(f)$.

Definition 1.10. Let $f : I \to J$ and consider $x \in I$. A point x is *eventually periodic of period* n if x is not periodic but there exists $m > 0$ such that $f^{n+i}(x) = f^i(x)$ for all $i \geq m$. That is, $f^i(x)$ is periodic for $i \geq m$.

Definition 1.11. Let $f : I \to J$ and consider $x, p \in I$. Let p be periodic of period n. A point x is *forward asymptotic* to p if $\lim_{i\to\infty} f^{in}(x) = p$. The stable set of p, denoted by $W^s(p)$, consists of all points foward asymptotic to p.

Definition 1.12. Let $f : I \to J$ and consider $x \in I$. A point x is a *critical point* of f if $f'(x) = 0$. The critical point is non-degenerate if $f''(x) \neq 0$. The critical point is degenerate if $f''(x) = 0$.

Definition 1.13. Let $f : I \to J$ and consider $x \in I$. The *forward orbit* of x is the set of points $x, f(x), f^2(x), ...$ and is denoted by $O^+(x)$. If f is a homeomorphism, we may define the *full orbit* of x, $O(x)$, as the set of points $f^n(x)$ for $n \in \mathbb{Z}$, and the *backward orbit* of x, $O^-(x)$, as the set of points $x, f^-1(x), f^-2(x)....$

In the next section, these definitions will be used to understand the path of hyperbolic points under multiple iterations.

1.2 Hyperbolic Points

The path of points under multiple iterations corresponds to the behavior of the map. In this section, we will specifically consider maps with hyperbolic periodic points because these are very important in dynamical systems.

Definition 1.14. Let $f : I \to J$. Let p be a periodic point of prime period n. The point p is *hyperbolic* if $|(f^n)'(p)| \neq 1$. The number $(f^n)'(p)$ is called the *multiplier* of the periodic point.

Proposition 1.15. *Let $f : I \to J$. Let p be a hyperbolic fixed point with $|f'(p)| < 1$. Then there is an open interval U about p such that if $x \in U$, then*

$$\lim_{n\to\infty} f^n(x) = p.$$

Proof. Let p be a hyperbolic fixed point with $|f'(p)| < 1$. Because f is continuous, there exists an $\epsilon > 0$ and an $A \in \mathbb{R}$ such that $|f'(x)| < A < 1$ for $x \in [p - \epsilon, p + \epsilon]$ or for x with $|x - p| < \epsilon$. p is a fixed point and therefore $f(p) = p$. Using the Mean Value Theorem we get the following inequality:

$$|f(x) - p| = |f(x) - f(p)| \leq A|x - p| < |x - p| < \epsilon.$$

So p is an accumulation point of $f(x)$.Therefore $|f(x) - p| < \epsilon$ or $f(x) \in [p - \epsilon, p + \epsilon]$. Notice that $|f^n(x)| < A^n < 1$ for all $n \in N$ and so $|f^n(x) - p| \leq A^n|x - p|$. It follows that $f^n(x) \to p$ as $n \to \infty$. Therefore there is an open interval $[p - \epsilon, p + \epsilon]$ about p such that if $x \in U$, then $\lim_{n\to\infty} f^n(x) = p$.

$\square$

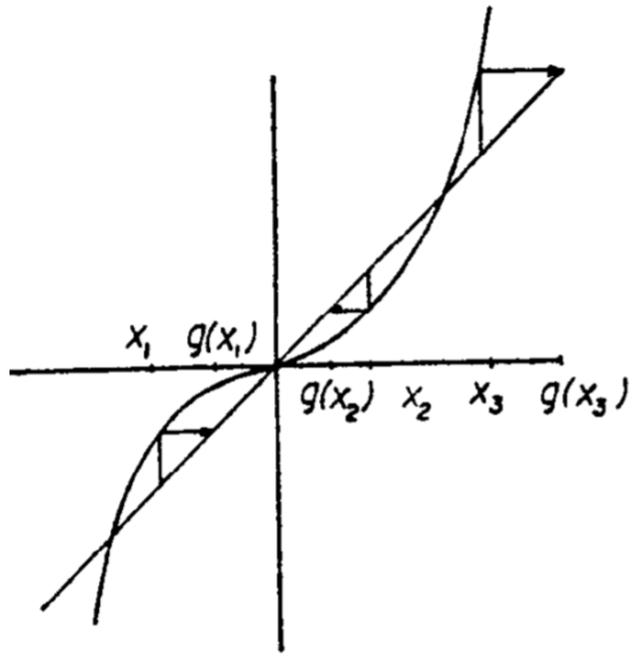

Figure 1.1.

Iteration map of $g(x) = x^3$.

In Figure 1.1, we can see the path of points under multiple iterations of this function. Because the map reflects the iterations about the line $y = x$, it is important to notice where the function intersects with this line. These intersections may be places where the behavior of a set of points changes. Here, we will define the different behaviors of the hyperbolic and fixed points.

Definition 1.16. Let $f : I \to J$. A hyperbolic point p of period n with $|(f^n)'(p)| < 1$ is called an *attracting fixed point(a attractor) or sink.*

Definition 1.17. Let $f : I \to J$. A fixed point p with $|f'(p)| > 1$ is called a *repelling fixed point(a repel*
or source.

It is clear that in Figure 1.1 $x = 0$ is an attracting periodic point and $x = -1$ and $x = 1$ are repelling points.

Proposition 1.18. *Let $f : I \to J$. Let p be a hyperbolic fixed point with $|f'(p)| > 1$. Then there is an open interval U of p such that, if $x \in U, x \neq p$, then there exists $k > 0$ such that $f^k(x) \notin U$.*

Proof. Let p be a hyperbolic fixed point with $|f'(p)| > 1$. Since $f \in C^1$, there exists $\epsilon > 0$ and an $A \in \mathbb{R}$ such that $|f'(p)| > A > 1$ and $|f'(x)| > A > 1$ for $x \in [p-\epsilon, p+\epsilon]$. Notice there exists an x where $x \neq p$ and $f'(x) > A$ so we can conclude that $\frac{|f(x)-f(p)|}{|x-p|} \geq A > 1$. Because we know $f(p) = p$, $|f(x)-f(p)| = |f(x)-p|$. We also know that $|f(x)-p| \geq A|x-p| > |x-p|$. Then $|f(x) - f(p)| > |x - p|$ so $\frac{|f(x)-f(p)|}{|x-p|} > 1$. Therefore, $f(x)$ does not approach p.

We then can use induction to prove $|f^n(x) - p| \geq A^n|x - p|$ for $n \geq 0$. Thus there exists $n \in \mathbb{N}$ such that $f^n(x) \notin U$. Then there is an open interval $U = [p - \epsilon, p + \epsilon]$ of p such that, if $x \in U$, $x \neq p$, then there exists $k > 0$ such that $f^k(x) \notin U$. □

The local behavior of hyperbolic periodic points can be determined by taking the derivative at that point. When the absolute value of the derivate changes from less than one to greater than one, it will intersect with the line $y = x$ and this is where we saw the behavior changed in Figure 1.1. In the next section, we will focus our attention on a specific quadratic function and how points behave in relation to it.

1.3 The Quadratic Family of Functions

In this section we will observe the behavior of $F_\mu(x) = \mu x(1 - x)$ as a dynamical system. This function will be essential to our understanding of chaos and is where my later studies branched from Devaney's book.

Proposition 1.19. *Let $F : I \to J$ and $\mu > 0$,*

(1) $F_\mu(0) = F_\mu(1) = 0$ and $F_\mu(p_\mu) = p_\mu$, where $p_\mu = \frac{\mu-1}{\mu}$.

(2) $0 < p_\mu < 1$ if $\mu > 1$.

Proposition 1.20. *Let $F : I \to J$. Suppose $\mu > 1$. If $x < 0$, then $F_\mu^n(x) \to -\infty$ as $n \to \infty$. If $x > 1$, then $F_\mu^n(x) \to -\infty$ as $n \to \infty$.*

Proof. Suppose $\mu > 1$. Let $x < 0$, then $\mu x(1-x) < x$ and $F_\mu(x) < x$. Because the first iteration of $F_\mu(x)$ is less than x we know it is a decreasing sequence. This sequence does not converge to p, a fixed point, because the magnitude of the derivative or slope of $f(x)$ increases as x decreases. Therefore, as x is reflected across the line $y = x$ it will travel away from $x = 0$. In conclusion, $F_\mu^n(x) \to -\infty$ as $n \to \infty$. Similarly we can see if $x > 1$ then $F_\mu(x) < 0$ so also $F_\mu^n(x) \to -\infty$.

□

Figure 1.2 illustrates Proposition 1.20. Notice that all of the necessary information when analyzing the behavior of this function will occur on the unit interval.

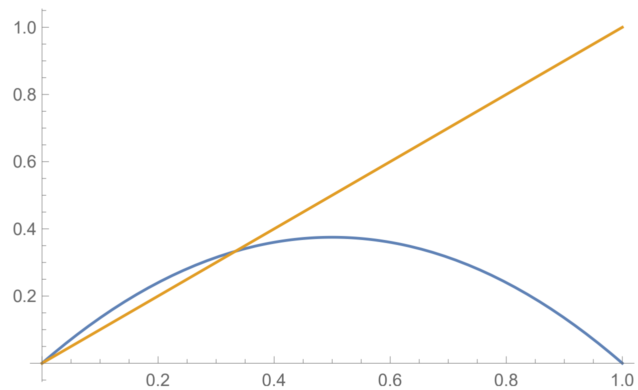

Figure 1.2.

Graph of $F_\mu(x) = \mu x(1-x)$ when $\mu = 1.5$.

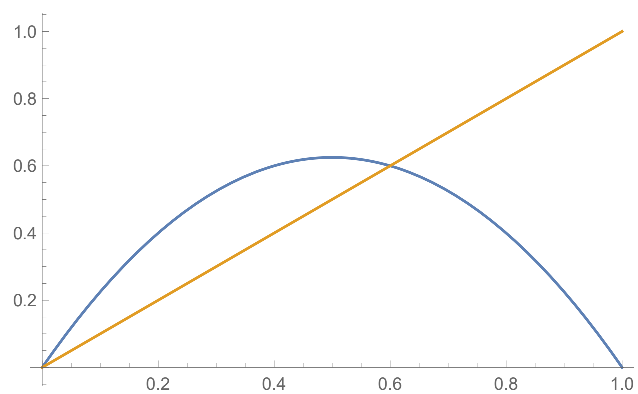

Figure 1.3.

Graph of $F_\mu(x) = \mu x(1-x)$ when $\mu = 2.5$.

Proposition 1.21. *Let $F : I \to J$ and let $1 < \mu < 3$.*

(1) F_μ has an attracting fixed point at $p_\mu = (\mu - 1)/\mu$ and a repelling fixed point at 0.

(2) If $0 < x < 1$, then

$$\lim_{n\to\infty} F_\mu^n(x) = p_\mu$$

.

Proof. Part 1 follows from Section 1.4 of Devaney's book. So F_μ has an attracting fixed point at $p_\mu = (\mu - 1)/\mu$ and a repelling fixed point at 0.

For part 2 we will consider three cases. First let $1 < \mu < 2$. Also, suppose $0 < x \leq 1/2$.

As you can see from Figure 1.2, above, $|F_\mu(x) - p_\mu| < |x - P_\mu|$ when $x \neq p_\mu$. Therefore, $F_\mu^n(x) \to p_\mu$ as $n \to \infty$ because it is an attracting fixed point.

Now let us consider $1/2 < x < 1$. From the graph we observe that $F_\mu(x)$ will be in the range $(0, 1/2)$ as well. Notice $F_\mu^n(x) = F_\mu^{n-1}(F_\mu(x))$. From the previous case we know $F_\mu(x) \to p_\mu$ and p_μ is a fixed point so we can conclude that $F_\mu^n(x) \to p_\mu$ as $n \to \infty$. Therefore $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$.

For our second case let $2 < \mu < 3$. As we can see from Figure 1.3 above all fixed points, p_μ are in between 1/2 and 1. For clarity let $\hat{p_\mu}$ represent the value in the interval $(0, 1/2)$ that is mapped onto p_μ by F_μ. We know $F - \mu^2$ maps the interval $[\hat{p_\mu}, p_\mu]$ into $[1/2, p_\mu]$. Therefore as we increase n $F_\mu^n(x)$ approaches p_μ. However, if $x < \hat{p_\mu}$ then we cannot simply apply the same argument. We must analyze the graph and pick a specific $k > 0$ such that $F_\mu^k(x) \in [\hat{p_\mu}, p_\mu]$. So, $F_\mu^{k+n}(x) \to p_\mu$ as $n \to \infty$. As a third case we can conclude F_μ maps the interval $(p_\mu, 1)$ onto $(0, p_\mu)$. We know $(0, 1) = (0, \hat{p_\mu}) \cup [\hat{p_\mu}, p_\mu] \cup (p_\mu, 1)$. Therefore $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$. Thirdly we will consider $\mu = 2$.

Because of these three cases we can conclude that $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$ for $1 < \mu < 3$.

$\square$

For Proposition 1.21, we see that F_μ has 2 fixed points in $[0,1]$ and all other points map to p_μ. Now we will define Λ. First let $A_1 = \{x \in I | F(x) \in \mathcal{A}_0\}$. For each $n \in \mathbb{N}$ let $A_n = \{x \in I | F^n(x) \in A_0\}$ $=\{x \in I | F^i(x) \in I$ for $I \leq n$ but $F^{n+1}(x) \notin I\}$. Then we can define $\Lambda = I - \left(\bigcup_{n=0}^{\infty} A_n\right)$ which consists of only the points which never escape from I. We can construct this set of points by successively removing open intervals, A_n, from the middle of closed sets, $I - \left(\bigcup_{n=0}^{n-1} A_n\right)$.

Theorem 1.22. *If $\mu > 2 + \sqrt{5}$, then Λ is a Cantor set.*

Proof. Suppose $\Lambda = I - \left(\bigcup_{n=0}^{\infty} A_n\right)$ where $A_n = \{x \in I | F^i(x) \in I$ for $i \leq n$ but $F^{n+1}(x) \notin I\}$. Let $\mu > 2 + \sqrt{5}$. There exists $\lambda > 1$ such that $|F'(x)| > \lambda$ for all $x \in \Lambda$. Using the chain rule we see that $|(F^n)'(x)| > \lambda^n$. Suppose Λ is not totally disconnected, so there exists $[x, y] \subset \Lambda$ such that $x \in \Lambda$ and $y \in \Lambda$ with $x \neq y$. Then $|(F^n)'(\alpha)| > \lambda^n$ for all $\alpha \in [x, y]$. Pick $n \in N$ such that $\lambda^n |y - x| > 1$. Applying the Mean Value Theorem we see that $|F^n(y) - F^n(x)| > \lambda^n |y - x| > 1$. So either $F^n(y)$ or $F^n(x)$ exists outside of I. This is a contradiction, so then Λ is totally disconnected, with no intervals. We can observe from the nature of how Λ was constructed that it is a nested intersection of closed intervals and therefore Λ is closed. Notice that every endpoint of A_k for some $k \in N$ is in Λ. Therefore Λ is a Cantor set.

$\square$

Definition 1.23. Let $f : I \to J$. A set $\Gamma \subset \mathbf{R}$ is a *repelling hyperbolic set for* f if Γ is closed, bounded and invariant under f and there exists an $N > 0$ such that $|(f^n)'(x)| > 1$ for all $n \geq N$ and all $x \in \Gamma$.

We can apply this definition to Λ when $\mu > 2 + \sqrt{5}$, then with $N = 1$ Λ is a repelling hyperbolic set. In the next section, we use this set to define dynamical structure of the quadratic map.

First, a symbolic model mapping will be constructed in order to define the dynamics as simply as possible.

Definition 1.24. Consider the space Σ_2 made up of sequences of $0's$ and $1's$. $\Sigma_2 = \{s = (s_0 s_1 s_2 ...) | s_j = 0 \text{ or } 1\}$.

Definition 1.25. Let s, t be points in Σ_2 such that $s = (s_0 s_1 s_2 ...)$ and $t = (t_0 t_1 t_2 ...)$. We define d as the distance between two points or for any $s, t \in \Sigma_2$

$$d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

Proposition 1.26. *d is metric on Σ_2.*

Proof. It follows $d[s,t] \geq 0$ for all $s, t \in \Sigma_2$. Also $d[s,t] = 0$ iff $s_i = t_i$ for all i. We know $|s_i - t_i| = |t_i - s_i|$ and so $d[s,t] = d[t,s]$. Let $r, s, t \in \Sigma_2$ then using the triangle inequality we get $|r_i - s_i| + |s_i - t_i| \geq |r_i - t_i|$. Then $d[r,s] + d[s,t] \geq d[r,t]$. In conclusion, d is metric on Σ_2.

□

In order to determine the distance between sets and which subsets of Σ_2 are closed and open, we need the metric, d.

Proposition 1.27. *Let $s, t \in \Sigma_2$ and suppose $s_i = t_i$, for $i = 0, 1, ..., n$. Then $d[s,t] \leq 1/2^n$. Conversely, if $d[s,t] < 1/2^n$, then $s_i = t_i$, for $i \leq n$.*

Proof. Let $s, t \in \Sigma_2$ and suppose $s_i = t_i$, for $i = 0, 1, ..., n$. If $s_i = t_i$ for $i \leq n$, then $d[s,t] = \sum_{i=0}^{n} \frac{|s_i - s_i|}{2^i} + \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}$. Therefore $d[s,t] \leq 1/2^n$.

Conversely, let $d[s,t] < 1/2^n$. If $s_j \neq t_j$ for some $j \leq n$ then $d[s,t] \geq \frac{1}{2^j} \geq \frac{1}{2^n}$. Therfore $d[s,t] < \frac{1}{2^n}$. So then $s_i = t_i$, for $i \leq n$.

□

Proposition 1.27 gives us a quick way to determine if two sets are close to one another.

Definition 1.28. The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is given by $\sigma(s_0 s_1 s_2 ...) = (s_1 s_2 s_3 ...)$.

Before we mention the next proposition it is important to note that if a subset U of S is *dense* in S then $\bar{U} = S$.

Proposition 1.29. *Let $\Sigma_2 = \{s = (s_0 s_1 s_2 ...) | s_j = 0$ or $1\}$ and the shift map $\sigma : \Sigma_2 \to \Sigma_2$.*

(1) The cardinality of $Per_n(\sigma)$ is 2^n.

(2) $Per(\sigma)$ is dense in Σ_2.

(3) There exists a dense orbit for σ in Σ_2.

Now that we have defined the shift map on Σ_2 in the next section we will compare it to the map on Λ.

1.5 How the Shift Map Relates to the Quadratic Map

In this section, we will show that the shift map on Σ_2 is essentially the same as the map F_μ on Λ. First denote the sets $I_0 \subset I$ such that I_0 contains the points on the left of A_0 and $I_1 \subset I$ such that I_1 contains the points on the right of A_0. It is important to notice whether the iterates of x fall in I_0 or I_1. This will give us a general idea of the behavior of the orbit.

Definition 1.30. Let $F : I \to J$. The *itinerary* of x is a sequence $S(x) = s_0 s_1 s_2 ...$ where $s_j = 0$ if $F_\mu^j(x) \in I_0$, $s_j = 1$ if $F_\mu^j(x) \in I_1$.

Theorem 1.31. *If $\mu > 2 + \sqrt{5}$, then $S : \Lambda \to \Sigma_2$ is a homeomorphism.*

Here is a part of the proof for Theorem 1.31. First let us define for all $n \in \mathbb{N}$, $I_{s_0 s_1 ... s_n} = \{x \in I | x \in I_{s_0}, F_\mu(x) \in I_{s_1}, ..., F_\mu^n(x) \in I_{s_n}\}$. We consider the preimage of a closed interval $A \subset I$. The preimage consists of 2 closed intervals, one being in I_0 and the other in I_1. We then looked at $I_{s_0 s_1 ... s_n}$ for all $n \in \mathbb{N}$ and determined it was a nested sequence of nonempty closed intervals. This helps show S is onto.

Theorem 1.32. *$S \circ F_\mu = \sigma \circ S$.*

Proof. Let $x \in \Lambda$ be defined by the nested sequence of intervals $\bigcap_{n \geq 0} I_{s_0 s_1 ...}$ determined by $S(x)$. So $I_{s_0 s_1 ... s_n} = I_{s_0} \cap F_\mu^{-1}(I_{s_1}) \cap ... \cap F_\mu^{-n}(I_{s_n})$. We also know $F_\mu(I_{s_0}) = I$. Then we can write $F_\mu(I_{s_0 s_1 ... s_n})$ as $I_{s_1} \cap F_\mu^{-1}(I_{s_2}) \cap ... F_\mu^{-n+1}(I_{s_n}) = I_{s_1 ... s_n}$. Then $SF_\mu(x) = SF_\mu(\cap_{n=0}^{\infty} I_{s_0 s_1 ... s_n}) = S(\cap_{n=0}^{\infty} I_{s_1 ... s_n}) = s_1 s_2 ... = \sigma S(x)$. Therefore $S \circ F_\mu = \sigma \circ S$.

□

Definition 1.33. Let $f : A \to A$ and $g : B \to B$ be two maps. f and g are said to be *topologically conjugate* if there exists a homeomorphism $h : A \to B$ such that, $h \circ f = g \circ h$. The homeomorphism h is called a topological conjugacy.

Since S is a homeomorphism and $S \circ F_\mu = \sigma \circ S$ we see σ and F_μ are topologically conjugate. Topologically conjugacy provides a way to test if two mappings have exactly the same dynamics. Because F_μ on Λ is topologically conjugate to the shift map we can say that the quadratic map has the characteristics from Proposition 1.29. In the following theorem we have written this proposition in terms of the quadratic map.

Theorem 1.34. *Let $F_\mu(x) = \mu x(1-x)$ such that $F : I \to J$ with $\mu > 2 + \sqrt{5}$. Then*

(1) The cardinality of $Per_n(F_\mu)$ is 2^n.

(2) $Per(F_\mu)$ is dense in Λ.

(3) F_μ has a dense orbit in Λ.

Topological conjugacy verifies that the shift map accurately models the quadratic map on its invariant set. In the next section we will consider if the quadratic map is chaotic or not.

1.6 Chaos

Many people define chaos differently but here we will address chaos from a topological approach.

Definition 1.35. Let f be the function such that $f : J \to J$ is said to be *topologically transitive* if for any pair of open sets $U, V \subset J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

Following this definition, we know a map on a compact metric space contains a dense orbit if and only if it is topologically transitive.

Definition 1.36. Let f be the function such that $f : J \to J$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that , for any $x \in J$ and any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

The following definition was chosen because it has become standard in some settings, i.e. topological dynamics, and it is very applicable to examples like the quadratic map.

Definition 1.37. Let V be a set. $f : V \to V$ is said to be *chaotic* on V if

(1) f has sensitive dependence on initial conditions.

(2) f is topologically transitive.

(3) periodic points are dense in V.

Another explanation of this definition is that a chaotic map cannot be broken down into different subsystems, unpredictably dependent on initial conditions and thirdly the periodic points are dense in V.

Definition 1.38. Let f be the function such that $f : J \to J$ is *expansive* if there exists $\nu > 0$ such that, for any $x, y \in J$, $x \neq y$, there exists n such that $|f^n(x) - f^n(y)| > \nu$.

The following two examples are the most noteworthy results we have gotten thus far. It uses all of the information obtained in the last 5 sections. Example 1.39 defines only a small subset of the unit interval. But Example 1.40 considers a larger chaotic region.

Example 1.39. The quadratic maps $F_\mu(x) = \mu x(1-x)$ are chaotic on Λ when $\mu > 2 + \sqrt{5}$.

Example 1.40. $F_4(x) = 4x(1-x)$ is chaotic on the interval $I = [0, 1]$.

Now that we have identified the quadratic map as chaotic under certain conditions we will change our function and determine if it too is chaotic.

CHAPTER TWO

New Function

I constructed a new family of functions by first setting constraints. I wanted the functions in this family to have two independent parameters and have three critical points, at $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$. I also wanted these functions to intersect with the x-axis at 0 and 1 always. Below is a family of functions that conforms to these constraints and that we chose to analyze. We also chose to focus on the parameters where and $0 \leq a < 1$ and $-1 < b < 0$.

$$f_{a,b}(x) = \begin{cases} 4ax & 0 \leq x < 0.25 \\ 4x(b-a) + 2a - b & 0.25 \leq x < 0.5 \\ 4x(a-b) + 3b - 2a & 0.5 \leq x < 0.75 \\ -4ax + 4a & 0.75 \leq x < 1 \end{cases} \tag{2.1}$$

Now the behavior of the functions in this family will heavily depend on the choice of the parameters a and b. While a and b are independent of one another, the combination of their values will characterize our functions outputs and graph. The parameter values a and b represent the $f(\frac{1}{4}) = f(\frac{3}{4})$ value and the $f(\frac{1}{2})$ respectively. If $b = -0.1$ and $a = 0.2$ the following figure is the corresponding graph.

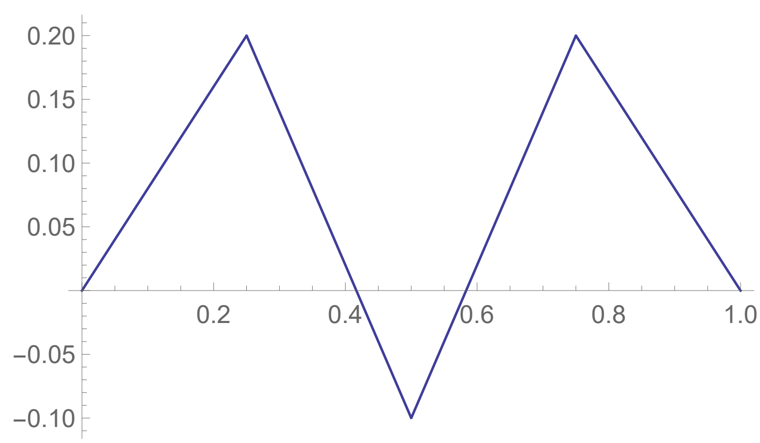

Figure 2.1.

Graph of $f_{0.2,-0.1}(x)$.

In order to find our critical points we took the derivative of our function, $f_{a,b}(x)$.

$$f'_{a,b}(x) = \begin{cases} 4a & 0 \leq x < 0.25 \\ 4b - a & 0.25 \leq x < 0.5 \\ 4a - b & 0.5 \leq x < 0.75 \\ -4a & 0.75 \leq x < 1 \end{cases} \tag{2.2}$$

As we notice from the following graph the derivative does not equal zero anywhere for this function $f_{0.2,-0.1}(x)$.But the derivative does not exist at our two maxima and one minimum points. Therefore, our function has critical points at $x = 0.25$ and $x = 0.5$ and $x = 0.75$ and one fixed point at $x = 0$.

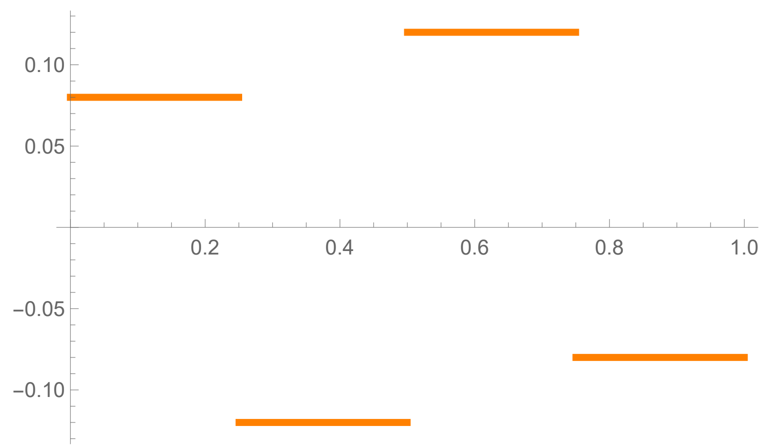

Figure 2.2.

Graph of $f'_{0.2,-0.1}(x)$.

If we had an unlimited amount of time we could test all the possible combinations of parameters values and ranges. Because our time was limited we confined a and b to the following ranges, $0 < a \leq 1$ and $-1 \leq b < 0$. More specifically, $b = -0.1$ for our initial analysis. This gives us a graph with two maxima and one minima and keeps our function outputs between -1 and 1. The following graphs show our function graphed alongside the line $y = x$ to easily show the intersections where fixed points are found.

The following graph is for the parameters $b = -0.1$ and $a = 0.15$. Notice the graph does not go above the line $y = x$. Therefore every point $x \in [0, b_0] \cup [b_1, 1]$ (where b_0 and b_1 are the x intercepts of our function excluding $x = 0, 1$) gets attracted to 0 as the function is iterated. The points $x \in (b_0, b_1)$ leave the unit interval after the first iteration.

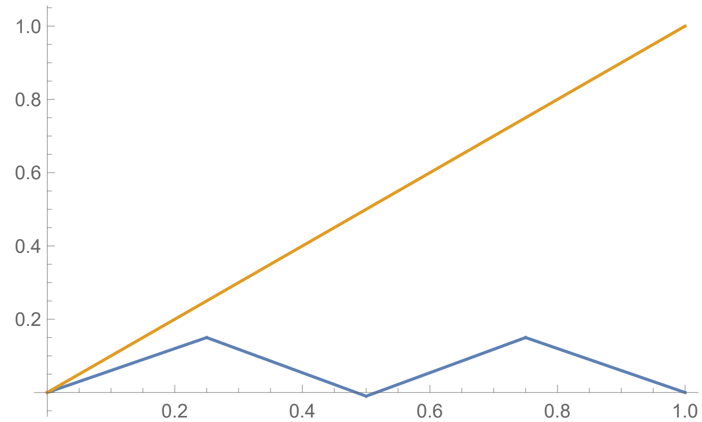

Figure 2.3.

Graph of $f_{0.15,-0.1}(x)$.

The following graph is for when $b = -0.1$ and $a = 0.25$. Notice the graph overlaps with the line $y = x$ from $0 \leq x \leq 0.25$. Therefore every point $x \in [0, 0.25]$ is a fixed point, but these points are not attracting fixed points because for $\epsilon > 0$ all points in $N_\epsilon(x)$ stay still instead of mapping closer to x. This is because these points all map directly to fixed points and then get stuck after a low number of iterations. The points $x \in (0.25, b_0] \cup [b_1, 1]$ mapped to $[0, 0.25]$. And again the points $x \in (b_0, b_1)$ leave the unit interval after the first iteration.

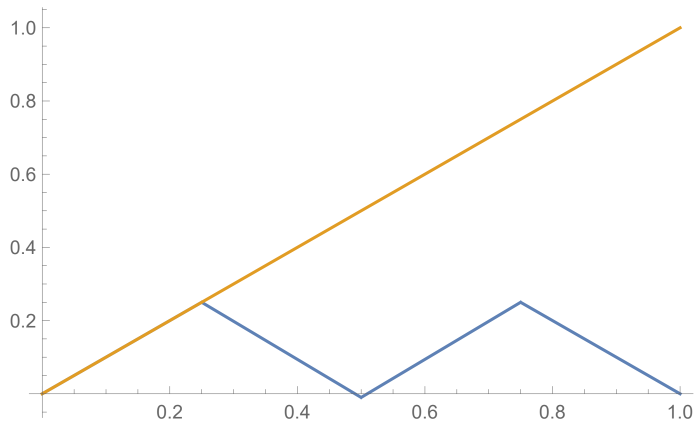

Figure 2.4.

Graph of $f_{0.25,-0.1}(x)$.

The following graph is for when $b = -0.1$ and $a = 0.5$. Notice the graph intersects with the line $y = x$ twice. Therefore we have two fixed points at $x = 0$ and $x = p_1$. We know the points $x \in (b_0, b_1)$ leave the unit interval after the first iteration. But the behavior of the rest of the points in the unit interval is for more complicated than our last 2 graphs. We will come back to this example in the next section.

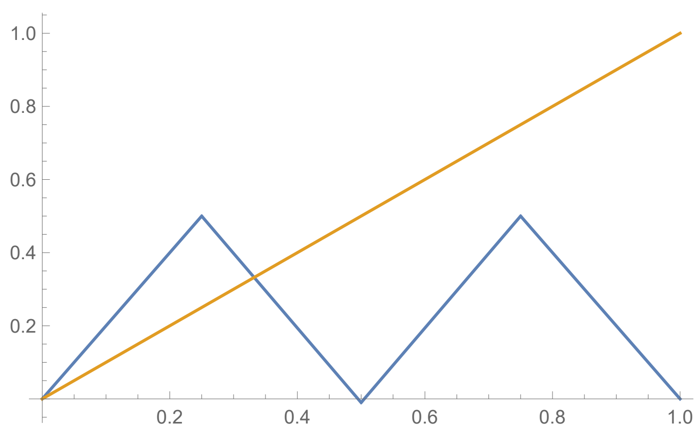

Figure 2.5.

Graph of $f_{0.5,-0.1}(x)$.

One method Devaney implemented to determine significant parameter values was using a bifurcation diagram. The following bifurcation diagrams have set $b = -0.1$ and are running iterations over many thousands of evenly spaced a values from 0 to 1. In Wolfram Mathematica I ran a program to calculate the bifurcation diagram of $f_{a,-0.1}$. The program calculated 500 iterations but only plotted the last 300 iterations. The horizontal axis is for the a values and the vertical axis is for the x values. The points and lines on this graph represent which x values move within the interval $[0, 1]$ at each specific iteration.

As you can see below there is a "drip" formed at $a = 0.25$ this makes sense following our interpretation of Figure 2.4 where we have fixed points that stay for $x \in [0, 0.25]$. Also the bifurcation diagram almost stops after $a \approx 0.452$, this means most x values leave $[0, 1]$ after 500 iterations when $a > 0.452$.

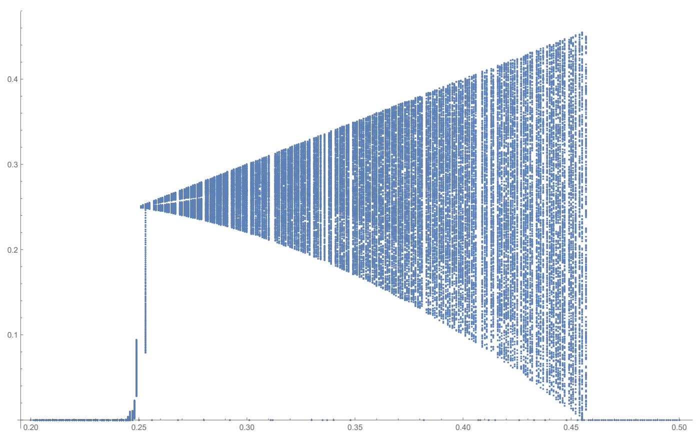

Figure 2.6.

Bifurcation diagram of $f_{a,-0.1}(x)$.

Here I have also included a zoomed in version of the bifurcation diagram to show there is an "eye-hole" in the diagram between $a = 0.25$ and $a = 0.3$. This "eye-hole" corresponds to x-values between 0.22 and 0.27. This means a relatively large interval of x values leaves after 500 iterations. The points that are left behind in $[0, 1]$ after many iterations will be essential in evaluating if the function is chaotic or not.

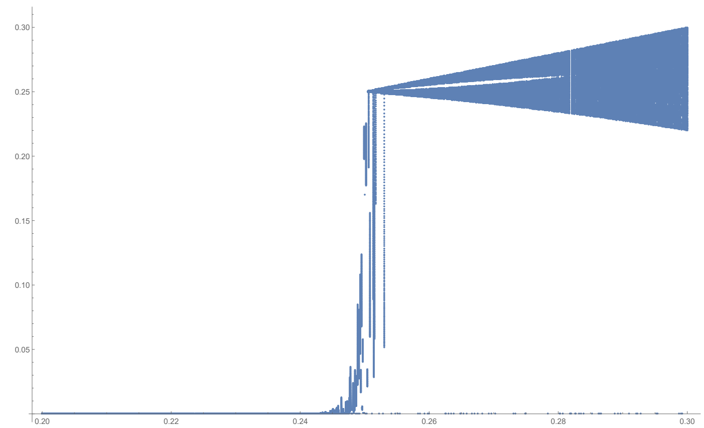

Figure 2.7.

Bifurcation zoomed in diagram of $f_{a,-0.1}(x)$.

In order to more specifically see what is going on after many iterations, I have included a series of drawings. Hopefully this will help us characterize the set of points which stay in $[0, 1]$.

For this first drawing $0.25 < a < 0.75$ because we have two fixed points, $x = 0$ and $x = p_1$ with $0.25 < p_1 < b_0$. We see that $x_1 < 0.25$ leaves I after two iterations and $x_2 > 0.75$ gets gets stuck in a periodic loop and stays in $[0, 1]$ under iteration.

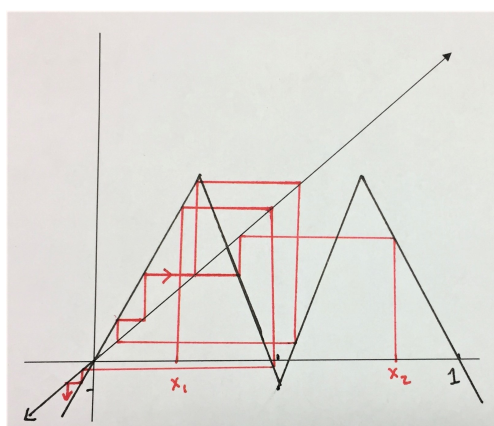

Figure 2.8.

Iteration map of $f_{0.5,-0.1}(x)$.

In Figure 2.9 we see than when $a = 0.75$ we have exactly three fixed points at $x = 0$, $x = p_1$ and $x = 0.75$ with $0.25 < p_1 < b_0$. Here x_1 gets stuck in a periodic loop and stays in I under iteration and x_2 leaves the unit interval after three iterations.

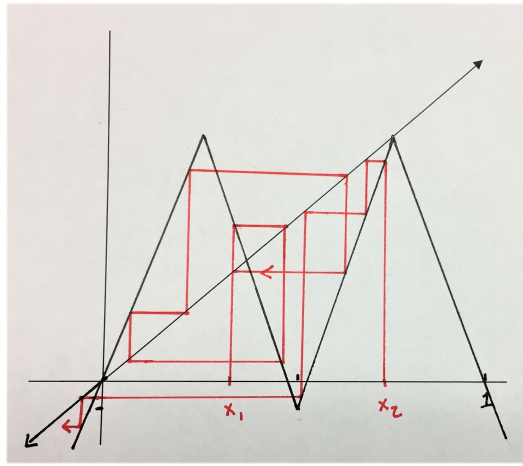

Figure 2.9.

Iteration map of $f_{0.75,-0.1}(x)$.

In the following drawing we see that when $0.75 < a < 1$ then we have four fixed points at $x = 0, x = p_1, x = p_2$ and $x = p_3$ with $0.25 < p_1 < b_0$, $b_1 < p_2 < 0.75$, $0.75 < p_3 < 1$. Here we see x_1 maps to a fixed point and stays in the unit interval permanently. However x_2 leaves I after six iterations.

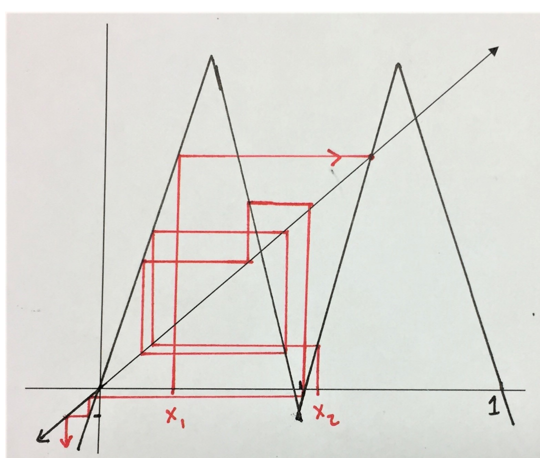

Figure 2.10.

Iteration map of $f_{0.9,-0.1}(x)$.

Now consider the properties of b_0 and b_1. If we fix $b = -0.1$ then as $a \to \infty$ $b_0 \to 0.5^+$ and $b_1 \to 0.5^-$, but $b_0, b_1 \neq 0.5$. And we calculated that $a = b_0$ when $a = b_0 \approx 0.4549$ and that $a = b_1$ when $a = b_1 \approx 0.5391$. Because of the bifurcation diagrams we know that when $a = 0.4549$ most of the x-values leave I after iterations.

In this final drawing we can see that when $0.4549 \approx b_0 < a < b_1 \approx 0.5391$ after the first iteration three intervals leave, (b_0, b_1), (x_1, x_2) and (x_3, x_4). And on the second iteration four more intervals will leave. If $a > b_1$ then five intervals would leave after one iteration and eight intervals would leave after the second iteration.

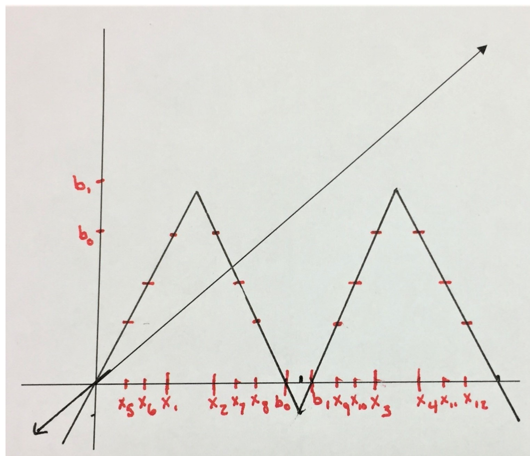

Figure 2.11.

Interval graph of $f_{a,-0.1}(x)$ with $0.4549 \approx b_0 < a < b_1 \approx 0.5391$.

2.2 Results

First we will prove that for $b = -0.1$ and $0.4549 \approx b_0 < a < b_1 \approx 0.5391$ the points remaining form a Cantor set, this proof will be similar to the proof of Theorem 1.22. First let us note $B_0 = (b_0, b_1)$ and $B_1 = (x_1, x_2) \cup (x_3, x_4)$ or we can define $B_n = \{x \in I | f^n_{a,b}(x) \in B_{n-1}\}$. So that $\bigcup_{n=0}^{\infty} B_n$ represents all of the points that eventually leave $[0, 1]$.

Theorem 2.1. *For $b = -0.1$ and $0.4549 \approx b_0 < a < b_1 \approx 0.5391$ the invariant set of $f_{a,b}(x)$ form a closed and totally disconnected set.*

Proof. We can prove that for $\Lambda = I - (\bigcup_{n=0}^{\infty} B_n)$:

(1) Λ is closed.

(2) Λ is totally disconnected.

Notice Λ is a nested intersection of closed intervals.

$I - (\bigcup_{n=0}^{\infty} B_n) = I \cap (\bigcup_{n=0}^{\infty} B_n)' = I \cap \bigcap_{n=0}^{\infty} (B_n)'$

B_n are open sets but the complements, $(B_n)'$, are closed sets. Also we know I is closed. So I $\cap \bigcap_{n=0}^{\infty} (B_n)'$ is closed and therefore Λ is a closed set.

Suppose $x, y \in \Lambda$ such that $[x, y] \subset \Lambda$. For $b = -0.1$ and $b_0 < a < b_1$ there exists a $\lambda > 1.6 > 1$ such that $|f'(x)| > \lambda$ for all $x \in \Lambda$. We also know by the chain rule $|(f^n)'(x)| > \lambda^n$. For all $\alpha \in [x, y]$ we know $|(f^n)'(\alpha)| > \lambda^n$. Pick $k \in \mathbb{N}$ such that $\lambda^k |y - x| > 1$. Applying the Mean Value Theorem we see that $|f^k(y) - f^k(x)| \geq \lambda^k |y - x| > 1$. So if this interval is contained in $\Lambda \subset [0, 1]$ then $f^k(y)$ or $f^k(x)$ is outside of $\Lambda \subset [0, 1]$, but we assumed the entire closed interval $[x, y]$ stayed in $\Lambda \subset [0, 1]$. This is a contradiction, so then Λ is totally disconnected and therefore has no intervals.

□

We conjecture that Λ is a perfect subset of I. If this is true Λ would be a Cantor set. This is because in order to prove $\Lambda = I - (\bigcup_{n=0}^{\infty} B_n)$ is a Cantor set we need to show:

(1) Λ is closed.

(2) Λ is totally disconnected.

(3) Λ is a perfect subset of I.

Finally we conjecture that our function $f_{a,b}(x)$ maps are chaotic on Λ when $b = -0.1$ and $0.4549 \approx b_0 < a < b_1 \approx 0.5391$. This is very similar to Example 1.39.

CHAPTER THREE

Conclusion

We set out to use Devaney's book on chaotic dynamical systems to help us analyze a multi-parameter family of functions. In the end we set constraints on our parameters and came to a conclusion about specific instances. Even though we were not able to address the family of functions in their entirety, we still discovered a sliver of their behavior.

Our final concluding discoveries are about the characteristics and behavior of our family of piecewise defined functions.

$$f_{a,b}(x) = \begin{cases} 4ax & 0 \leq x < 0.25 \\ 4x(b-a) + 2a - b & 0.25 \leq x < 0.5 \\ 4x(a-b) + 3b - 2a & 0.5 \leq x < 0.75 \\ -4ax + 4a & 0.75 \leq x < 1 \end{cases} \tag{3.1}$$

We know if $|a| > 1$ and $|b| > 1$ then all points except $x = 0$ leave the unit interval after multiple iterations. Let b_0 and b_1 be the points in $[0, 1]$ with $f_{a,b}(b_0) = 0 = f_{a,b}(b_1)$. We know if $a = 0.25$ and $-\infty < b \leq 0.5$ there are two cases. If $b < 0$ then $x \in [0, b_0] \cup [b_1, 1]$ stay in the unit interval. An the second case is if $b \geq 0$ then all $x \in [0, 1]$ stay in $[0, 1]$ under iteration. Then if $b_0 < a < b_1$ and $-1 \leq b < 0$ then the points remaining in I form a closed and totally disconnected set. We conjecture that this invariant set is also a perfect subset of I, making the set itself a Cantor set. Also we conjectured that under these constraints f maps are chaotic on its invariant set.

3.1 Future Work

As I said in Results Section, I conjecture that the invariant set of $f_{a,b}(x)$ is a perfect subset of I and I also conjecture that this set would be chaotic under the specified parameter values. These conjectures have not been thoroughly proved. Future work would include perfecting and solidifying these assumptions. Also, because we did not consider all cases and combinations of parameter values, there is more work to do in order to completely analyze this multi-parameter family of functions' behavior. Cases such as $a < 0$ and $0 < b \leq 1$ were not considered. Some of these cases were not considered because they seemed less interesting after an initial judgment. Also some more future work is needed to determine the significance of chaos in our family of functions. We identified an occurrence but not a meaning or an application.

As I am headed to graduate school for biostatistics, I am very interested in ways that pure math concepts can be applied to real world problems. Hopefully I can find a way to apply the knowledge gained from this project to some future research. I am hoping to get involved in a project with the JFK Medical Center in Nashville, TN, specifically with their studies on mental disabilities.

Chaotic dynamical systems have been applied to many fields including electrical and computer engineering when trying to program simulations, music theory when determining the resonance and frequency of chords, and to the medical field specifically within the studies of neuron stimulation. The more mathematicians understand and discover in chaotic dynamical systems, the more possibilities for technological advancement and scientific discovery in the future.

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